B.Sc. Physics Part-III

Mathematical Physics

10/09/2020

Solution of Laplace's equation:

Laplace's equation is given by $\nabla^2 u = 0$ — ①

Let us consider the two dimensional case,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{——} \quad ②$$

Let the solution of equation ② is

$$u = X(x)\,Y(y) \quad \text{—} \quad ③$$

Substituting eq. ③ in eq. ②, we obtain

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

or $$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

or $$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} \quad \text{——} \quad ④$$

x and y are independent variable. Therefore, equation ④ holds good only if each side of the equation is equal to a constant.

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k$$

and $$\frac{1}{Y}\frac{d^2Y}{dy^2} = k$$

or $\frac{d^2X}{dx^2} - kX = 0$

and $\frac{d^2Y}{dy^2} + kY = 0$

The Solution of above equation is given by the following conditions

~~X = a₁e~~

(i) when k = +ve and equal to α^2;

$X = a_1 e^{\alpha x} + a_2 e^{-\alpha x}$ and $Y = a_3 \cos\alpha y + a_4 \sin\alpha y$

(ii) when k is negative and equal to $-\alpha^2$;

$X = a_5 \cos\alpha x + a_6 \sin\alpha x$; $Y = a_7 e^{\alpha y} + a_8 e^{-\alpha y}$

(iii) when $k = 0$;

$X = a_9 x + a_{10}$; $Y = a_{11} y + a_{12}$

Thus the all possible Solution is given by

$u = (a_1 e^{\alpha x} + a_2 e^{-\alpha x})(a_3 \cos\alpha y + a_4 \sin\alpha y)$

$u = (a_5 \cos\alpha x + a_6 \sin\alpha x)(a_7 e^{\alpha y} + a_8 e^{-\alpha y})$

$u = (a_9 x + a_{10})(a_{11} y + a_{12})$

• From the all possible Solutions we consider that solution which is consistent with the given boundary conditions.

H.W. Solve the Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to the conditions $u(0,y) = u(l,y) = u(x,0) = 0$ and $u(x,a) = \sin n\pi x/l$.